Box 2.3 DIFFERENTIALS

The "exterior derivative" or "gradient" $\boldsymbol{d}f$ of a function f is a more rigorous version of the elementary concept of "differential."

In elementary textbooks, one is presented with the differential df as representing "an infinitesimal change in the function $f(\mathcal{P})$" associated with some infinitesimal displacement of the point $\mathcal{P}$; but one will recall that the displacement of $\mathcal{P}$ is left arbitrary, albeit infinitesimal. Thus df represents a change in f in some unspecified direction.

But this is precisely what the exterior derivative $\boldsymbol{d}f$ represents. Choose a particular, infinitesimally long displacement $\boldsymbol{v}$ of the point $\mathcal{P}$. Let the displacement vector $\boldsymbol{v}$ pierce $\boldsymbol{d}f$ to give the number $\langle \boldsymbol{d}f, \boldsymbol{v}\rangle = \partial_{\boldsymbol{v}} f$. That number is the change of f in going from the tail of $\boldsymbol{v}$ to its tip. Thus $\boldsymbol{d}f$, before it has been pierced to give a number, represents the change of f in an unspecified direction. The act of piercing $\boldsymbol{d}f$ with $\boldsymbol{v}$ is the act of making explicit the direction in which the change is to be measured. The only failing of the textbook presentation, then, was its suggestion that $\boldsymbol{d}f$ was a scalar or a number; the explicit recognition of the need for specifying a direction $\boldsymbol{v}$ to reduce $\boldsymbol{d}f$ to a number $\langle \boldsymbol{d}f, \boldsymbol{v}\rangle$ shows that in fact $\boldsymbol{d}f$ is a 1-form, the gradient of f.

§2.8. THE CENTRIFUGE AND THE PHOTON

Vectors, metric, 1-forms, functions, gradients, directional derivatives: all these geometric objects and more are used in flat spacetime to represent physical quantities; and all the laws of physics must be expressible in terms of such geometric objects.

Geometric objects in action: example of centrifuge and photon

As an example, consider a high-precision redshift experiment that uses the Mössbauer effect (Figure 2.9). The emitter and the absorber of photons are attached to

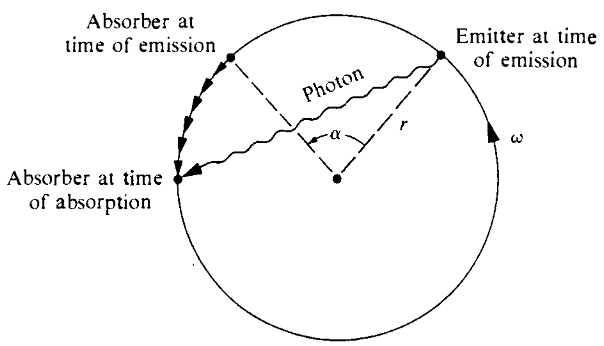

Figure 2.9.
The centrifuge and the photon.

the rim of a centrifuge at points separated by an angle α, as measured in the inertial laboratory. The emitter and absorber are at radius r as measured in the laboratory, and the centrifuge rotates with angular velocity ω. PROBLEM: What is the redshift measured,

$$z = (\lambda_{\text{absorbed}} - \lambda_{\text{emitted}})/\lambda_{\text{emitted}},$$

in terms of ω, r, and α?

SOLUTION: Let $\boldsymbol{u}_e$ be the 4-velocity of the emitter at the event of emission of a given photon; let $\boldsymbol{u}_a$ be the 4-velocity of the absorber at the event of absorption; and let $\boldsymbol{p}$ be the 4-momentum of the photon. All three quantities are vectors defined without reference to coordinates. Equally coordinate-free are the photon energies E_e and E_a measured by emitter and absorber. No coordinates are needed to describe the fact that a specific emitter emitting a specific photon attributes to it the energy E_e; and no coordinates are required in the geometric formula

$$E_e = -\boldsymbol{p} \cdot \boldsymbol{u}_e \tag{2.29}$$

for E_e. [That this formula works can be readily verified by recalling that, in the emitter's frame, $u_e{}^0 = 1$ and $u_e{}^j = 0$; so

$$E_e = -p_\alpha u_e{}^\alpha = -p_0 = +p^0$$

in accordance with the identification "(time component of 4-momentum) = (energy."] Analogous to equation (2.29) is the purely geometric formula

$$E_a = -\boldsymbol{p} \cdot \boldsymbol{u}_a$$

for the absorbed energy.

The ratio of absorbed wavelength to emitted wavelength is the inverse of the energy ratio (since $E = h\nu = hc/\lambda$):

$$\frac{\lambda_a}{\lambda_e} = \frac{E_e}{E_a} = \frac{-\boldsymbol{p} \cdot \boldsymbol{u}_e}{-\boldsymbol{p} \cdot \boldsymbol{u}_a}.$$

This ratio is most readily calculated in the inertial laboratory frame

$$\frac{\lambda_a}{\lambda_e} = \frac{p^0 u_e{}^0 - p^j u_e{}^j}{p^0 u_a{}^0 - p^j u_a{}^j} = \frac{p^0 u_e{}^0 - \boldsymbol{p} \cdot \boldsymbol{u}_e}{p^0 u_a{}^0 - \boldsymbol{p} \cdot \boldsymbol{u}_a}. \tag{2.30}$$

(Here and throughout we use boldface Latin letters for three-dimensional vectors in a given Lorentz frame; and we use the usual notation and formalism of three-dimensional, Euclidean vector analysis to manipulate them.) Because the magnitude of the ordinary velocity of the rim of the centrifuge, $v = \omega r$, is unchanging in time, $u_e{}^0$ and $u_a{}^0$ are equal, and the magnitudes—but not the directions—of $\boldsymbol{u}_e$ and $\boldsymbol{u}_a$ are equal:

$$u_e{}^0 = u_a{}^0 = (1 - v^2)^{-1/2}, \ |\boldsymbol{u}_e| = |\boldsymbol{u}_a| = v/(1 - v^2)^{1/2}.$$

From the geometry of Figure 2.9, one sees that $\boldsymbol{u}_e$ makes the same angle with $\boldsymbol{p}$ as does $\boldsymbol{u}_a$. Consequently, $\boldsymbol{p} \cdot \boldsymbol{u}_e = \boldsymbol{p} \cdot \boldsymbol{u}_a$, and $\lambda_{\text{absorbed}}/\lambda_{\text{emitted}} = 1$. *There is no redshift!*

Notice that this solution made no reference whatsoever to Lorentz transformations—they have not even been discussed yet in this book! The power of the geometric, coordinate-free viewpoint is evident!